

# SHADOWING, EXPANSIVENESS AND STABILITY OF DIVERGENCE-FREE VECTOR FIELDS

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ABSTRACT. Let  $X$  be a divergence-free vector field defined on a closed, connected Riemannian manifold. In this paper, we show the equivalence between the following conditions:

- $X$  is in the  $C^1$ -interior of the set of *expansive* divergence-free vector fields.
- $X$  is in the  $C^1$ -interior of the set of divergence-free vector fields which satisfy the *shadowing property*.
- $X$  is in the  $C^1$ -interior of the set of divergence-free vector fields which satisfy the *Lipschitz shadowing property*.
- $X$  has no singularities and  $X$  is *Anosov*.

## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let  $M$  be an  $n$ -dimensional,  $n \geq 3$ , closed, connected and smooth Riemannian manifold, endowed with a volume form, which has associated a measure  $\mu$ , called the Lebesgue measure, and let  $d$  denote the Riemannian distance. Let  $\mathfrak{X}^s(M)$  be the set of vector fields and let  $\mathfrak{X}_\mu^s(M)$  be the set of divergence-free vector fields, both defined on  $M$  and endowed with the  $C^s$  Whitney topology,  $s \geq 1$ . From now on, we consider  $s = 1$ . A vector field  $X$  has associated a flow, denoted by  $X^t$ ,  $t \in \mathbb{R}$ . Denote by  $Per(X)$  the union of the *closed orbits* of  $X$  and by  $Sing(X)$  the union of the *singularities* of  $X$ . A subset of  $M$  is said to be *regular* if it has no singularities. Denote by  $Crit(X)$  the set of the closed orbits and the singularities associated to  $X$ . A singularity  $p$  is *linear* if there exist smooth local coordinates around  $p$  such that  $X$  is linear and equal to  $DX(p)$  in these coordinates (see [17, Definition 4.1]).

Take a  $C^1$ -vector field and a regular point  $x$  in  $M$  and let  $N_x := X(x)^\perp \subset T_x M$  denote the  $(\dim(M) - 1)$ -dimensional normal bundle of  $X$  at  $x$  and  $N_{x,r} = N_x \cap \{u \in T_x M : \|u\| < r\}$ , for  $r > 0$ . Since, in general,  $N_x$  is not  $DX_x^t$ -invariant, we define the *linear Poincaré flow*

$$P_X^t(x) := \Pi_{X^t(x)} \circ DX_x^t,$$

where  $\Pi_{X^t(x)} : T_{X^t(x)}M \rightarrow N_{X^t(x)}$  is the canonical orthogonal projection.

Let  $\Lambda$  be a compact,  $X^t$ -invariant and regular set. If  $N_\Lambda$  admits a  $P_X^t$ -invariant splitting  $N_\Lambda = N_\Lambda^s \oplus N_\Lambda^u$ , such that there is  $\ell > 0$  satisfying

$$\|P_X^\ell(x)|_{N_x^s}\| \leq \frac{1}{2} \text{ and } \|P_X^{-\ell}(X^\ell(x))|_{N_{X^\ell(x)}^u}\| \leq \frac{1}{2},$$

for any  $x \in \Lambda$ , we say that  $\Lambda$  is *hyperbolic*. A vector field  $X$  is said to be *Anosov* if the whole manifold  $M$  is hyperbolic. Let  $\mathcal{A}_\mu^1(M)$  denote the set of Anosov  $C^1$ -divergence-free vector fields.

Take  $T > 0$  and  $\delta > 0$ . A map  $\psi : \mathbb{R} \rightarrow M$  is a  $(\delta, T)$ -pseudo-orbit of a flow  $X^t$  if, for any  $\tau \in \mathbb{R}$ ,  $d(X^t(\psi(\tau)), \psi(\tau + t)) < \delta$ , for any  $|t| \leq T$ .

Take  $\epsilon > 0$ . A pseudo-orbit  $\psi$  of a flow  $X^t$  is  $\epsilon$ -shadowed by some orbit of  $X^t$  if there is  $x \in M$  and an increasing homeomorphism  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ , called reparametrization, which satisfies  $\alpha(0) = 0$  and such that  $d(X^{\alpha(t)}(x), \psi(t)) < \epsilon$ , for every  $t \in \mathbb{R}$ .

**Definition 1.1.** A  $C^1$ -vector field  $X$  satisfies the shadowing property if for any  $\epsilon > 0$  there is  $\delta > 0$  such that any  $(\delta, T)$ -pseudo-orbit  $\psi$ , for  $T > 0$ , is  $\epsilon$ -shadowed by some orbit of  $X$ . Let  $\mathcal{S}^1(M)$  and  $\mathcal{S}_\mu^1(M)$  denote the sets of vector fields in  $\mathfrak{X}^1(M)$  and  $\mathfrak{X}_\mu^1(M)$ , respectively, satisfying the shadowing property.

In the mid 1990's (see [14]) it was shown that a dissipative diffeomorphism in the  $C^1$ -interior of the set of diffeomorphisms with the shadowing property is structurally stable. More recently, Lee and Sakai (see [7]) proved that if  $X \in \text{int } \mathcal{S}^1(M)$  and has no singularities then  $X$  satisfies the Axiom A and the strong transversality conditions, where  $\text{int } S$  stands for the  $C^1$ -interior of a set  $S \subset \mathfrak{X}^1(M)$ .

Now, we introduce a weaker definition.

**Definition 1.2.** A  $C^1$ -vector field  $X$  satisfies the Lipschitz shadowing property if there are positive constants  $\ell$  and  $\delta_0$  such that any  $(\delta, T)$ -pseudo-orbit  $\psi$ , with  $T > 0$  and  $\delta \leq \delta_0$  is  $\ell\delta$ -shadowed by an orbit of  $X$ . Let  $\mathcal{LS}^1(M)$  and  $\mathcal{LS}_\mu^1(M)$  denote the sets of vector fields in  $\mathfrak{X}^1(M)$  and  $\mathfrak{X}_\mu^1(M)$ , respectively, satisfying the Lipschitz shadowing property.

It is immediate, from the previous definitions, that  $\mathcal{LS}^1(M) \subset \mathcal{S}^1(M)$  and  $\mathcal{LS}_\mu^1(M) \subset \mathcal{S}_\mu^1(M)$ . In [16], Tikhomirov proved that, for dissipative vector fields, Lipschitz shadowing is equivalent to structural stability. Recently, Pilyugin and Tikhomirov proved the same result for dissipative diffeomorphisms (see [12]). We can find in [11] the proof of that Anosov vector fields satisfy the Lipschitz shadowing property.

Let us now present the notion of *expansive vector field*.

**Definition 1.3.** A  $C^1$ -vector field  $X$  is *expansive* if for any  $\epsilon > 0$  there is  $\delta > 0$  such that if  $d(X^t(x), X^{\alpha(t)}(y)) \leq \delta$ , for all  $t \in \mathbb{R}$ , for  $x, y \in M$  and a continuous map  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  with  $\alpha(0) = 0$ , then  $y = X^s(x)$ , where  $|s| \leq \epsilon$ . Denote by  $\mathcal{E}^1(M) \subset \mathfrak{X}^1(M)$  the set of expansive vector fields and by  $\mathcal{E}_\mu^1(M) \subset \mathfrak{X}_\mu^1(M)$  the set of divergence-free expansive vector fields, both endowed with the  $C^1$  Whitney topology.

In 1970's, Mañé proved that if a dissipative diffeomorphism  $f$  is in the  $C^1$ -interior of the set of expansive diffeomorphisms then  $f$  is Axiom A and satisfies the quasi-transversality condition (see [8]). Later, in [9], Moriyasu, Sakai and Sun proved the same result for dissipative vector fields. Moreover, they proved that if  $X \in \text{int } \mathcal{E}^1(M)$  and has the shadowing property then  $X$  is Anosov. Recently, Pilyugin and Tikhomirov proved that an expansive dissipative diffeomorphism having the Lipschitz shadowing property is Anosov (see [12]).

In this article, we intend to characterize divergence-free vector fields, with a topological property of Anosov systems, such as topological stability under  $C^1$ -open conditions: shadowing and expansiveness. We prove the following:

**Theorem 1.** *For the divergence-free setting, one has that*

$$\text{int } \mathcal{E}_\mu^1(M) = \text{int } \mathcal{S}_\mu^1(M) = \text{int } \mathcal{LS}_\mu^1(M) = \mathcal{A}_\mu^1(M).$$

## 2. DEFINITIONS AND AUXILIARY RESULTS

In this section, we state some definitions and present some results that will be used in the proofs.

Let  $\Lambda$  be a compact,  $X^t$ -invariant and regular set. Consider a splitting  $N = N^1 \oplus \cdots \oplus N^k$  over  $\Lambda$ , for  $1 \leq k \leq n-1$ , such that all the subbundles have constant dimension. This splitting is *dominated* if it is  $P_X^t$ -invariant and there exists  $\ell > 0$  such that, for every  $0 \leq i < j \leq k$  and every  $x \in \Lambda$ , one has

$$\|P_X^\ell(x)|_{N_x^i}\| \cdot \|P_X^{-\ell}(X^\ell(x))|_{N_{X^\ell(x)}^j}\| \leq \frac{1}{2}, \quad \forall x \in \Lambda.$$

The following result can be obtained following the ideas presented in [4, Proposition 2.4].

**Theorem 2.1.** *Let  $X \in \mathfrak{X}_\mu^1(M)$  and let  $\mathcal{U}$  be a small  $C^1$ -neighbourhood of  $X$ . Then, for any  $\epsilon > 0$ , there exist  $l, \tau > 0$  such that, for any  $Y \in \mathcal{U}$  and any closed orbit  $x$  of  $Y^t$  of period  $\pi(x) > \tau$ ,*

- *either  $P_Y^t$  admits an  $l$ -dominated splitting over the  $Y^t$ -orbit of  $x$*

- or else for any neighbourhood  $U$  of  $x$ , there exists an  $\epsilon$ - $C^1$ -perturbation  $\tilde{Y}$  of  $Y$ , coinciding with  $Y$  outside  $U$  and along the orbit of  $x$ , such that  $P_{\tilde{Y}}^{\pi(x)}(x) = \text{id}$ , where  $\text{id}$  denotes the identity on  $N_x$ .

To prove Theorem 1, we also need to state the definition of *star vector field*.

**Definition 2.1.** A  $C^1$ -vector field  $X$  is a *star vector field* if there exists a  $C^1$ -neighborhood  $\mathcal{U}$  of  $X$  in  $\mathfrak{X}^1(M)$  such that if  $Y \in \mathcal{U}$  then every point in  $\text{Crit}(Y)$  is hyperbolic. Moreover, a  $C^1$ -divergence-free vector field  $X$  is a *divergence-free star vector field* if there exists a  $C^1$ -neighborhood  $\mathcal{U}$  of  $X$  in  $\mathfrak{X}_\mu^1(M)$  such that if  $Y \in \mathcal{U}$  then every point in  $\text{Crit}(Y)$  is hyperbolic. The set of star vector fields is denoted by  $\mathcal{G}^1(M)$  and the set of divergence-free star vector fields is denoted by  $\mathcal{G}_\mu^1(M)$ .

Accordingly with this definition, in [6, Theorem 1] it is proved the following result.

**Theorem 2.2.** If  $X \in \mathcal{G}_\mu^1(M)$  then  $\text{Sing}(X) = \emptyset$  and  $X$  is Anosov.

A 3-dimensional proof of this result is presented in [5] and a version for 4-dimensional symplectic Hamiltonian vector fields can be found in [3].

The following result says that the linear Poincaré flow cannot admit a dominated splitting over the set of regular points of  $M$  if the vector field has a linear hyperbolic singularity of saddle-type.

**Proposition 2.3.** [17, Proposition 4.1] If  $X \in \mathfrak{X}^1(M)$  admits a linear hyperbolic singularity of saddle-type then  $P_X^t$  does not admit any dominated splitting over  $M \setminus \text{Sing}(X)$ .

A vector field  $X$  is *topologically mixing* if, given any nonempty open sets  $U, V \subset M$ , there is  $T > 0$  such that, for any  $t \geq T$ , we have  $X^t(U) \cap V \neq \emptyset$ . We end this section with a result stating that,  $C^1$ -generically, the divergence-free vector fields are topologically mixing.

**Theorem 2.4.** [2, Theorem 1.1] There exists a  $C^1$ -residual subset  $\mathcal{R} \subset \mathfrak{X}_\mu^1(M)$  such that if  $X \in \mathcal{R}$  then  $X$  is a topologically mixing vector field.

### 3. PROOF OF THE THEOREM

**Lemma 3.1.** If  $X \in \text{int}\mathcal{E}_\mu^1(M)$  then any closed orbit of  $X$  is hyperbolic.

*Proof.* Take  $X \in \text{int}\mathcal{E}_\mu^1(M)$  and  $\mathcal{U}$  a  $C^1$ -neighbourhood of  $X$  in  $\mathcal{E}_\mu^1(M)$ . Let  $p$  be a point in a closed orbit of  $X$  with period  $\pi > 0$  and  $U_p$  a

small neighbourhood of  $p$  in  $M$ . By contradiction, assume that there is an eigenvalue  $\lambda$  of  $P_X^\pi(p)$  such that  $|\lambda| = 1$ .

Applying Zuppa's Theorem (see [18]), we can find  $Y \in \mathcal{U}$  such that  $Y \in \mathfrak{X}_\mu^\infty(M)$ ,  $Y^\pi(p) = p$  and  $P_Y^\pi(p)$  has an eigenvalue  $\lambda$  with  $|\lambda| = 1$ .

**Remark 3.1.** Notice that if  $P_Y^\pi(p)$  has not an eigenvalue  $\lambda$  with  $|\lambda| = 1$ , it has an eigenvalue  $\tilde{\lambda}$  such that  $|\tilde{\lambda}| \approx 1$ . So, we just have to perform a  $C^1$ -conservative perturbation  $\bar{Z}$  of  $Y$ , by [4, Lemma 3.2], such that  $P_{\bar{Z}}^\pi(p)$  has an eigenvalue  $\bar{\lambda}$  with  $|\bar{\lambda}| = 1$ .

Accordingly with Moser's Theorem (see [10]), there is a smooth conservative change of coordinates  $\varphi_p : U_p \rightarrow T_p M$  such that  $\varphi_p(p) = \vec{0}$ . Let  $f_Y : \varphi_p^{-1}(N_p) \rightarrow \Sigma$  be the Poincaré map associated to  $Y^t$ , where  $\Sigma$  denotes the Poincaré section through  $p$ , and take  $\mathcal{V}$  a  $C^1$ -neighbourhood of  $f_Y$ .

By [4, Lemma 3.2], taking  $\mathcal{T}$  a small flowbox of  $Y^{[0, t_0]}(p)$ ,  $0 < t_0 < \pi$ , we have that there are  $Z \in \mathcal{U}$ ,  $f_Z \in \mathcal{V}$  and  $\epsilon > 0$  such that:  $Z^t(p) = Y^t(p)$ ,  $t \in \mathbb{R}$ ;  $P_Z^{t_0}(p) = P_Y^{t_0}(p)$ ;  $Z|_{\mathcal{T}^c} = Y|_{\mathcal{T}^c}$  and

$$f_Z(x) = \begin{cases} \varphi_p^{-1} \circ P_Y^\pi(p) \circ \varphi_p(x) & , x \in B_{\epsilon/4}(p) \cap \varphi_p^{-1}(N_p) \\ f_Y(x) & , x \notin B_\epsilon(p) \cap \varphi_p^{-1}(N_p). \end{cases}$$

Notice that  $P_Z^\pi(p)$  still has an eigenvalue  $\lambda$  with  $|\lambda| = 1$ .

Since  $Z \in \mathcal{E}_\mu^1(M)$ , for a sufficiently small  $\epsilon > 0$ , there is  $0 < \delta < \epsilon$  such that if  $d(Z^t(x), Z^{\alpha(t)}(y)) \leq \delta$ , for any  $t \in \mathbb{R}$ ,  $x, y \in M$  and  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  continuous such that  $\alpha(0) = 0$ , then  $y = Z^s(x)$ , where  $|s| \leq \epsilon$ .

Take  $0 < \delta' < \delta$  such that if  $x, y \in M$  satisfy  $d(x, y) < \delta'$  then  $d(Z^t(x), Z^t(y)) < \delta$ , for  $0 \leq t \leq \pi$ .

Firstly, assume that  $\lambda = 1$  and fix the associated non-zero eigenvector  $v$  such that  $\|v\| < \delta'$ . Take  $\varphi_p^{-1}(v) \in \varphi_p^{-1}(N_p) \setminus \{p\}$  and note that

$$f_Z(\varphi_p^{-1}(v)) = \varphi_p^{-1} \circ P_Y^\pi(p) \circ \varphi_p(\varphi_p^{-1}(v)) = \varphi_p^{-1} \circ P_Y^\pi(p)(v) = \varphi_p^{-1}(v).$$

So,  $d(p, \varphi_p^{-1}(v)) = d(p, f_Z(\varphi_p^{-1}(v))) = \|v\| < \delta'$ . Then, as was mentioned before,  $d(Z^t(p), Z^t(\varphi_p^{-1}(v))) < \delta$ , for  $0 \leq t \leq \pi$ . Therefore, we can find a continuous function  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ , with  $\alpha(0) = 0$ , such that  $d(Z^t(p), Z^{\alpha(t)}(\varphi_p^{-1}(v))) < \delta$ , for every  $t \in \mathbb{R}$ . Now, since  $Z \in \mathcal{E}_\mu^1(M)$ ,  $\varphi_p^{-1}(v) = Z^s(p)$ , for  $|s| \leq \epsilon$ . This is a contradiction, because  $\varphi_p^{-1}(v) \in \varphi_p^{-1}(N_p) \setminus \{p\}$ .

Now, if  $|\lambda| = 1$  but  $\lambda \neq 1$ , we point out that, by [4, Lemma 3.2], we can find  $W \in \mathcal{U}$  such that  $P_W^\pi(p)$  is a rational rotation. Then, there is  $T \neq 0$  such that  $P_W^{T+\pi}(p) = id$ . So, we can go on with the previous

argument in order to reach the same contradiction. So, any closed orbit of  $X$  is hyperbolic.  $\square$

**Lemma 3.2.** *If  $X \in \text{int } \mathcal{S}_\mu^1(M)$  then any closed orbit of  $X$  is hyperbolic.*

*Proof.* Take  $X \in \text{int } \mathcal{S}_\mu^1(M)$ ,  $\mathcal{U}$  a  $C^1$ -neighbourhood of  $X$  in  $\mathcal{S}_\mu^1(M)$  and  $p$  be a closed orbit of  $X$  with period  $\pi > 0$ . By contradiction, assume that there is an eigenvalue  $\lambda$  of  $P_X^\pi(p)$  such that  $|\lambda| = 1$ .

By Zuppa's Theorem (see [18]), we can find  $Y \in \mathcal{U}$  such that  $Y \in \mathfrak{X}_\mu^\infty(M)$ ,  $Y^\pi(p) = p$  and  $P_Y^\pi(p)$  has an eigenvalue  $\lambda$  with  $|\lambda| = 1$ , as we remarked in the proof of Lemma 3.1.

Consider  $\varphi$  and  $Z \in \mathcal{U}$  as described in the proof of Lemma 3.1 and

$$f_Z(x) = \begin{cases} \varphi_p^{-1} \circ P_Y^\pi(p) \circ \varphi_p(x) & , x \in B_{\epsilon_0}(p) \cap \varphi_p^{-1}(N_p) \\ f_Y(x) & , x \notin B_{4\epsilon_0}(p) \cap \varphi_p^{-1}(N_p), \end{cases}$$

where  $\epsilon_0 > 0$  is small.

As it was explained in the proof of Lemma 3.1, we can assume  $\lambda = 1$  and fix the associated non-zero eigenvector  $v$  such that  $\|v\| = \epsilon_0/2$ . Define  $\mathcal{I}_v = \{sv : 0 \leq s \leq 1\}$ .

Since  $Z \in \mathcal{S}_\mu^1(M)$ , for any  $\epsilon > 0$  there is  $\delta > 0$  such that every  $(\delta, T)$ -pseudo-orbit is  $\epsilon$ -shadowed by some orbit  $y$  of  $Z^t$ , for  $T > 0$ . Fix  $0 < \epsilon < \frac{\epsilon_0}{4}$ . The idea now is to construct a  $(\delta, T)$ -pseudo-orbit of  $Z^t$ , adapting the strategy described on [7, Proposition A]. Let us present the highlights of that proof.

Let  $x_0 = p$  and  $t_0 = 0$ . Since  $p$  is a parabolic closed orbit, we construct a finite sequence  $\{(x_i, t_i)\}_{i=0}^I$ , where  $I \in \mathbb{N}$ ,  $t_i > 0$ ,  $x_i \in \varphi_p^{-1}(\mathcal{I}_v)$ , for  $1 \leq i \leq I$ , such that:  $x_I = \varphi_p^{-1}(v)$ ;  $d(Z^t(f_Z(x_i)), Z^t(x_{i+1})) < \delta$ , for  $|t| \leq T$  and  $0 \leq i \leq I-1$ ;  $Z^{t_i}(x_i) = f_Z(x_i)$ . So, letting  $S_n = \sum_{i=0}^n t_i$ , for  $0 \leq n \leq I$ , the map  $\psi : \mathbb{R} \rightarrow M$  defined by

$$\psi(t) = \begin{cases} Z^t(x_0) & , t < 0 \\ Z^{t-S_n}(x_{n+1}) & , S_n \leq t < S_{n+1}, 0 \leq n \leq I-2 \\ Z^{t-S_{I-1}}(x_I) & , t \geq S_{I-1}. \end{cases}$$

is a  $(\delta, T)$ -pseudo-orbit of  $Z^t$ . Now, since  $Z \in \mathcal{U}$ , there is a reparametrization  $\alpha$  and a point  $y \in B_\epsilon(p) \cap \varphi_p^{-1}(N_{p,\epsilon})$  which  $\epsilon$ -shadows  $\psi$ , that is,  $d(Z^{\alpha(t)}(y), \psi(t)) < \epsilon$ , for any  $t \in \mathbb{R}$ . Note that, since  $\lambda = 1$ ,

$$d(x_0, x_I) = d(p, \varphi_p^{-1}(v)) = d(p, f_Z(\varphi_p^{-1}(v))) = \|v\| = \frac{\epsilon_0}{2} > 2\epsilon.$$

But, since  $Z$  has the shadowing property,

$$d(x_0, x_I) \leq d(x_0, Z^{\alpha(S_{I-1})}(y)) + d(Z^{\alpha(S_{I-1})}(y), \psi(S_{I-1})) < 2\epsilon,$$

which is a contradiction.  $\square$

**Lemma 3.3.** *If  $X \in \mathfrak{X}_\mu^1(M)$  has a singularity then, for any neighbourhood  $\mathcal{V}$  of  $X$ , there is an open and nonempty set  $\mathcal{U} \subset \mathcal{V}$  such that any  $Y \in \mathcal{U}$  has a linear hyperbolic singularity.*

*Proof.* Let  $p$  be a singularity of  $X \in \mathfrak{X}_\mu^1(M)$  and  $\epsilon > 0$ . By a small  $C^1$ -conservative perturbation of  $X$  (see [4]), we can find  $X_1$ ,  $\epsilon$ - $C^1$ -close to  $X$ , with a hyperbolic singularity  $p$ . Denote by  $\mathcal{V}$  a  $C^1$ -neighbourhood of  $X_1$  in  $\mathfrak{X}_\mu^1(M)$  where the analytic continuation of  $p$  is well-defined. Now, by Zuppa's Theorem (see [18]), there is a smooth vector field  $X_2 \in \mathcal{V}$  with a hyperbolic singularity  $p_2$ . If the eigenvalues of  $DX_2(p_2)$  satisfy the nonresonance conditions of the Sternberg linearization theorem (see [15]) then there is a smooth diffeomorphism conjugating  $X_2$  and its linear part around  $p_2$ . If the nonresonance conditions are not satisfied then we can perform a  $C^1$ -conservative perturbation of  $X_2$ , so that the eigenvalues satisfy the nonresonance conditions. So, since the set of divergence-free vector fields satisfying the nonresonance conditions is an open and dense set in  $\mathfrak{X}_\mu^1(M)$ , there is a  $C^1$ -neighbourhood  $\mathcal{U}$  of  $X_2$  in  $\mathcal{V}$  such that any vector field  $X_3 \in \mathcal{U}$  is conjugated to its linear part, meaning that  $X_3$  has a linear hyperbolic singularity.  $\square$

*Proof of Theorem 1.* Take  $X \in \text{int}\mathcal{E}_\mu^1(M)$  and let  $\mathcal{U}$  be a  $C^1$ -neighbourhood of  $X$  in  $\mathcal{E}_\mu^1(M)$ , small enough such that Theorem 2.1 holds.

Recall that a conservative version of Pugh and Robinson's *General Density Theorem* (see [13]) asserts that,  $C^1$ -generically, the closed orbits are dense in  $M$ . Denote by  $\mathcal{PR}_\mu^1(M)$  the Pugh and Robinson's residual set in  $\mathfrak{X}_\mu^1(M)$  and by  $\mathcal{R}$  the residual set given by Theorem 2.4.

By contradiction, assume that there is  $p \in \text{Sing}(X)$ . By Lemma 3.3, there is  $Y \in \mathcal{U} \cap \mathcal{R} \cap \mathcal{PR}_\mu^1(M)$  such that  $p \in \text{Sing}(Y)$  is linear hyperbolic, and so of saddle-type. So, by Proposition 2.3,  $P_Y^t$  does not admit any dominated splitting over  $M \setminus \text{Sing}(Y)$ .

We point out that, by Lemma 3.1, any closed orbit of  $Y$  is hyperbolic. Now, as in the proof of [6, Lemma 3.1], take a closed orbit  $x$  of  $Y$  with arbitrarily large period. So, by Theorem 2.1, there are constants  $\ell, \tau > 0$  such that  $P_Y^t$  admits an  $\ell$ -dominated splitting over the  $Y^t$ -orbit of  $x$  with period  $\pi(x) > \tau$ . Since  $Y \in \mathcal{R}$ , by the volume preserving Arnaud Closing Lemma (see [1, p.13]), there is a sequence of vector fields  $Y_n \in \mathcal{U} \cap \mathcal{R}$ ,  $C^1$ -converging to  $Y$ , and, for every  $n \in \mathbb{N}$ ,  $Y_n$  has a closed orbit  $\gamma_n = \gamma_n(t)$  of period  $\pi_n$  such that  $\lim_{n \rightarrow \infty} \gamma_n(0) = x$  and  $\lim_{n \rightarrow \infty} \pi_n = +\infty$ . Therefore, by Theorem 2.1,  $P_{Y_n}^t$  admits an  $\ell$ -dominated splitting over the orbit  $\gamma_n$ , for large  $n$ . Choosing  $i \in J \subseteq \mathbb{N}$ , there is



a sequence of  $Y_i$  with  $P_{Y_i}^t$  having an  $\ell$ -dominated splitting on a closed orbit  $p_i$  and such that the dimensions of the invariant bundles do not

depend on  $i$ . Then, given that  $M = \limsup_n \gamma_n = \bigcap_{N \in \mathbb{N}} \left( \bigcup_{n \geq N}^\infty \gamma_n \right)$ , we

prove that  $P_Y^t$  admits a dominated splitting over  $M \setminus \text{Sing}(Y)$ . But this is a contradiction. So,  $\text{Sing}(X) = \emptyset$  and, by Lemma 3.1, one has that if  $X \in \text{int } \mathcal{E}_\mu^1(M)$  then  $X \in \mathcal{G}_\mu^1(M)$ . Then, by Theorem 2.2,  $X$  is Anosov.

Now, take  $X \in \text{int } \mathcal{S}_\mu^1(M)$ . Applying Lemma 3.2, we can follow an analogous strategy to that one described above and prove that if  $X \in \text{int } \mathcal{S}_\mu^1(M)$  then  $\text{Sing}(X) = \emptyset$  and  $X$  is Anosov.

In order to conclude the proof of Theorem 1, it is enough to see that  $\mathcal{LS}_\mu^1(M) \subset \mathcal{S}_\mu^1(M)$  and that  $\mathcal{A}_\mu^1(M) \subset \mathcal{LS}_\mu^1(M)$ , by [11, Theorem 1.5.1].  $\square$

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